Scand. Actuarial J. 1975: 129–144

Some Problems in Hadwiger Fertility Graduation

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Abstract

The Hadwiger function has been recommended for the analytic graduation of fertility curves. In its original parametrization, this graduating function has some rather problematic aspects in terms of parameter identifiability and interpretability. This can be remedied by a suitable reparametrization. However, a second practical identification problem arises insofar as the curves of the Hadwiger density and the gamma density with similar parameter values turn out to be virtually interchangeable in a number of numerical examples. Our impression is that the Hadwiger density is a bit more flexible than the gamma density. In this respect, therefore, the Hadwiger density seems to have a slight edge on the gamma density. These are the main concerns of this paper. In addition some findings are presented on methods of parameter estimation in current use. Moment type estimators turn out to be seriously inconsistent in several actual cases, and their replacement by minimum chi-square estimators is recommended.

1. Introduction

1 A. The diagram of age-specific fertility rates for a population, based on data for a calendar period, say, will typically picture a curve which looks much like a left-skewed probability density, but with superposed fluctuations. A number of functions have been suggested as a basis for the analytic graduation of such a diagram, the gamma and beta densities being particularly popular. Another possibility is the Hadwiger density. Empirical studies, both by ourselves and by others, have shown this to be a flexible function which gives a good fit to data from a wide range of populations. This paper presents some findings on the use of the latter density as a graduating function. It is part of the documentation of a larger project. Further empirical and theoretical results have been presented elsewhere (Hoem et al., 1974; Berge & Hoem, 1974).

1 B. Let us briefly sketch the practical background of the present study before we go into its details. The authors have investigated regional fertility in Norway around 1970. The 449 communes of the country as of January 1, 1971, have been grouped into 77 "fertility regions", mainly on the basis of their industrial composition, their geographic location, and whether they are characterized by systematic in- or out-migration. Each region will typically have a total resident

population of some 30 to 50 thousand inhabitants. For each region, age-specific fertility rates have been calculated from data for the period of 1968–1971, and the Hadwiger density has been fitted to the corresponding fertility curve.

By means of the analytic graduation, the description of regional age-specific fertility can be reduced to stating the value of a low-dimensional parameter vector for each region. Our hope is that subsequently these regional parameter vector values can be related to the values of ecological variables describing various aspects of the corresponding communities.

1 C. We have found that the four-parameter version of the Hadwiger function suggested independently by Yntema (1969) and Gilje (1969), and described again in Section 2 below, gives a very good fit to all of our 77 fertility curves, as judged by such criteria as least squares and eye-ball inspection of curve plots. This confirms similar findings by Yntema and Gilje.

[For references to the literature on fertility graduation up to about 1970, see Subsection 1.2 in Hoem (1972). Other contributions have been given by Brass (1968), Hunyadi & Szakolczai (1970), Mitra (1970), Gilje & Yntema (1971), Murphy & Nagnur (1972), Farid (1973), Romaniuk (1973), Mitra & Romaniuk (1973), Norman & Gilje (1973), and Hyrenius et al. (1974).]

1 D. During our work with this part of the project, we have discovered some problematic aspects of the use of the Hadwiger density for fertility graduation. One of us (Berge) has constructed cases where widely different values of the four original Hadwiger function parameters produce curves which are so close to each other as to be practically indistinguishable. Demographic theory does not provide a basis sufficient for choosing among the cases which have been uncovered in the graduation of real data, to say nothing about hedging against parameter value combinations which we do not know about. This means that although we are reasonably satisfied with the way in which the Hadwiger *curves* represent the level and age-pattern of fertility, we know of no way to "invert" this representation and get an interpretable and tolerably unique set of values for the Yntema–Gilje *parameters* of the Hadwiger function.

1 E. To overcome this difficulty, we have tried one of the obvious routes open to us, viz., we have sought a different parametrization of the Hadwiger function. We have taken the position that what is important, is a good fit of the curves, and we have looked for a set of four stable parameters, i.e., parameters whose values will change appreciably only when the Hadwiger curve changes in some important way. (It has not appeared possible to get a generally satisfactory curve fit with less than four parameters.) At the same time, we have wanted parameters which could be given natural demographic interpretations.

It has seemed to us that descriptive measures which are standard in any characterization of a skew bell-shaped curve, would fit our bill. For our parameters, therefore, we have selected the mode, the mean, and the variance of the Hadwiger density, as well as the area under the fertility curve. This is not such a sensational choice, and *post hoc* one may perhaps feel that we could have discovered these parameters without going through the mechanics of analytic graduation. This detour has served three purposes, however. First, it has assured us that we would need four parameters, and not some other number; and it has convinced us that these four ones are sufficient for our purposes. Secondly, the introduction of the values of these parameters into the graduating function provides us with a complete schedule of fertility rates for all ages, a fact which is useful in many circumstances, such as in connection with population projections. [Mitra & Romaniuk (1973) and Romaniuk (1973) stress the latter point.] Thirdly, analytic graduation furnishes a better set of parameter estimators than the naive ones as soon as the graduating function has been specified.

1 F. The latter point is not without its own problems, however. For comparison, we have used the gamma density as a basis for a second analytic graduation of a number of the Norwegian regional fertility curves mentioned above. The fit is roughly as satisfactory as the one given by the Hadwiger function, though the latter may possibly be a bit more flexible. This replicates earlier findings by others. We have also fitted a Hadwiger function to the graph of a gamma density, and vice versa. In all cases investigated, the Hadwiger function fits the gamma curve nicely. Similarly, the gamma function fits the Hadwiger curve well too. In both cases, the fit is occasionally somewhat less than excellent. The gamma density and the Hadwiger density seem practically interchangeable as graduating functions, at least in the cases we have studied. Thus, we are up against a second identification problem, viz., in our choice of a graduating function.

1 G. Most previous investigations have used moment type methods to fit functions to the observed fertility curves, and have not applied techniques like least squares or minimum chi-square. [Gilje (1969), Gilje & Yntema (1971), and Norman & Gilje (1974) are notable exceptions.] For the case of Hadwiger graduation, parameter estimators based on moments, the mode, etc., have been suggested by Yntema (1969), and some of their statistical properties have been discussed by Hoem (1972, Subsection 7.4).

Moment type estimators frequently have the nice feature that they can be represented by relatively simple mathematical formulas. This is not so with procedures like least squares, which will usually involve some iterative numerical method of function minimization (except in special cases, like when the graduating function is linear in its parameters). Such aspects may be part of the explanation why the more involved methods have been less popular. In our computerized age, however, the numerical work involved should be much less of a problem than it was previously, and we feel that there are good reasons, both intuitive and theoretical, to prefer minimum chi-square techniques, say, to moment methods. For one thing, a minimum chi-square or least squares criterion will frequently be used to measure the goodness-of-fit even of a graduation based on a moment method. For another, Hoem (1972) has proved that minimum chi-square methods provide estimators which are consistent and have uniformly minimum variance, and he has established the optimality of these estimators on other criteria as well (Hoem et al., 1974). Furthermore, he has pointed out (Hoem, 1972, page 591, Remark 8) that Yntema's estimators for the parameters of the Hadwiger function may be inconsistent, i.e., they may not converge in probability to the parameters estimated when the population size increases. Our numerical findings reported below show that such inconsistency can be considerable. This problem is inherent in the moment method approach to graduation, so that it is not particular to the Yntema estimators, nor is it particular to Hadwiger graduation for that matter. Indeed, we replicate this finding with tree sets of parameter values for the gamma density. In our opinion, this is a serious argument against relying solely upon such estimators.

Moment methods do have some practical interest, however, in that they can provide the starting values which are necessary for any iterative estimation procedure. The starting values can be important for the speed of convergence and other aspects of the iterative procedure, so that the statistical properties of the estimators which produce the starting values are by no means unimportant. In our experience, estimators such as those suggested by Yntema are quite sufficiently accurate for providing starting values, and we have been unable to improve upon them. Also, the problem of inconsistency seems to be less important for stable parameters than for others.

1 H. It may be of some interest to learn that we have found O'Neill's implementation of Nelder & Mead's simplex algorithm for function minimization (Nelder & Mead, 1965; O'Neill, 1971) better suited to our purposes of graduation than the Fletcher–Powell algorithm implemented by Gruvaeus & Jöreskog (1970).

2. The Hadwiger function

2 A. Let

$$f_H(x) = (H/\sqrt{\pi})\, x^{-3/2} \exp\{-H^2(x + x^{-1} - 2)\} \quad \text{for } x > 0. \tag{2.1}$$

Then f_H is a probability density on the positive real axis. We shall call it the Hadwiger density, and shall call H the Hadwiger parameter. (We follow the tradition of using capital Latin letters for the parameters in Hadwiger graduation.) The corresponding cumulant generating function is

$$\phi_H(s) = \ln \int_0^\infty e^{sx} f_H(x)\, dx = 2H^2 \left\{1 - \left(1 - \frac{s}{H^2}\right)^{1/2}\right\}.$$

This distribution has a mean of 1, a variance of $1/(2H^2)$, and its kth cumulant is

$$\varkappa_k = \frac{(2k-3)(2k-5)\dots 3\times 1}{(2H^2)^{k-1}} \quad \text{for } k \geqslant 2. \tag{2.2}$$

Thus, the measures of skewness and curtosis are

$$\gamma_1 = \frac{\varkappa_3}{\varkappa_2^{3/2}} = \frac{3}{\sqrt{2}H} \quad \text{and} \quad \gamma_2 = \frac{\varkappa_4}{\varkappa_2^2} = \frac{15}{2H^2}.$$

The density is unimodal, and its mode is at

$$Z = \frac{(1+16H^4/9)^{1/2}-1}{4H^2/3}. \tag{2.3}$$

We note that $Z<1$ for all H, so that the mode is always less than the mean here.

[Formula (2.2) corrects the erroneous formula (7.24) previously given by Hoem (1972) as well as a misprint in formula (7) in Yntema (1956).]

2B. Hadwiger (1940) suggested using the function

$$h_1(x) = \frac{R}{T} f_H\left(\frac{x}{T}\right) \quad \text{for } x>0,$$

to graduate age-specific fertility rates, x representing age of mother at childbearing. (This particular parametrization is actually due to Yntema, 1952.) The area under the curve of h_1 is R, which is taken to represent the gross reproduction rate or the total fertility rate, according as only girl babies or both boy and girl babies are included in the count of the liveborn. The parameter T is taken as the mean age at childbearing, and it will typically have a value somewhere between 25 and 30 in human populations. (For a discussion of the concept of the mean age at childbearing, see Hoem, 1971.)

No one seems to have succeeded in giving convincing demographic interpretation of the Hadwiger parameter H. The most useful relation it appears in, may be

$$S^2 = \frac{T^2}{2H^2}, \tag{2.4}$$

where S^2 is the variance of the density $h_1(\cdot)/R$. For all graduations known to Yntema in 1961, H had a value slightly above 3, a fact which inspired him into calling it a demometric invariant (Yntema, 1961). (Largely, this finding has been upheld in later work.)

In Fig. 1, we have plotted the function h_1 with $R=1$ and $T=27$ for a number of values of H. With the normal type of values for the parameters, the curve of h_1 will cling to the x-axis up to some age in the early teens, whence it will start to raise its head, reach its mode at an abscissa of ZT (which will be $0.92T$ for $H=3$, or 24.8 when $T=27$), and then subside again to become very small for $x>50$.

2C. Hadwiger & Ruchti (1941) and Yntema (1952, 1953, 1956) fitted this function to their satisfaction to a number of fertility curves, but Tekse (1967) discovered that it gave a very bad fit to some important age-patterns of fertility. Yntema (1969) and Gilje (1969) then independently introduced a fourth parameter, which we shall call D, so that the graduating function took the following form:

$$h(x; R, H, T, D) = \frac{R}{T} f_H\left(\frac{x-D}{T}\right) \quad \text{for } x > D. \tag{2.5}$$

Empirical studies have shown this function to be highly flexible and to give a good fit to an extensive set of fertility age-patterns. (Some of its limitations have been explored by Yntema (1969) and by Gilje & Yntema (1971).)

2D. It is not easy, however, to give the parameters in (2.5) a reasonable demographic interpretation. The area R retains its previous interpretation as a measure of the fertility level, of course, but the mean and modal ages of childbearing now become

$$U = D + T \quad \text{and } M = D + TZ, \tag{2.6}$$

respectively, so that T loses its previous interpretation as the mean, and Z loses its previous interpretation as the proportion of the mean which constitutes the mode. The disturbing element is the new parameter D whose substantive meaning is problematic. Although $h(x; R, H, T, D)$ is positive for all $x > D$, D does not have such an interpretation as the lowest fertile age, say, since the value of $h(x; R, H, T, D)$ can be negligible for a considerable interval above D. Witness the description for $D = 0$ at the end of Section 2B. In Table 1, we shall even give realistic examples where D is negative.

We had difficulty in interpreting the Hadwiger parameter H in (2.4) already, and this does not become easier in (2.5). It is not a demometric invariant any more; its value can vary widely from one graduation to the next. It still satisfies (2.4) with S^2 equal to the variance of $h(\cdot; R, H, T, D)/R$.

2E. There does not seem to be any guidance to be had from the numerical values of the parameters in empirical fits either. In fact one of us (Berge) has discovered that there are combinations of values of H, T, and D which seem completely wild as compared to what we are used to seeing in empirical graduations, yet the corresponding curves seem quite reasonable as fertility curves go. Six such combinations are given in Table 1, along with corresponding values of U, M, and S^2. Figs. 2 and 3 contain plots of the corresponding curves.

These plots bring out another interesting feature too, viz., that even though curves 1, 2 and 3 have widely different values of H, T, and D, their graphs are quite close to each other. Similarly for curves 4, 5 and 6. Normed sums of squares of deviations between pairs of curves in each group have been listed in Table 2. (A discussion of the norming procedure is given in Subsection 2F

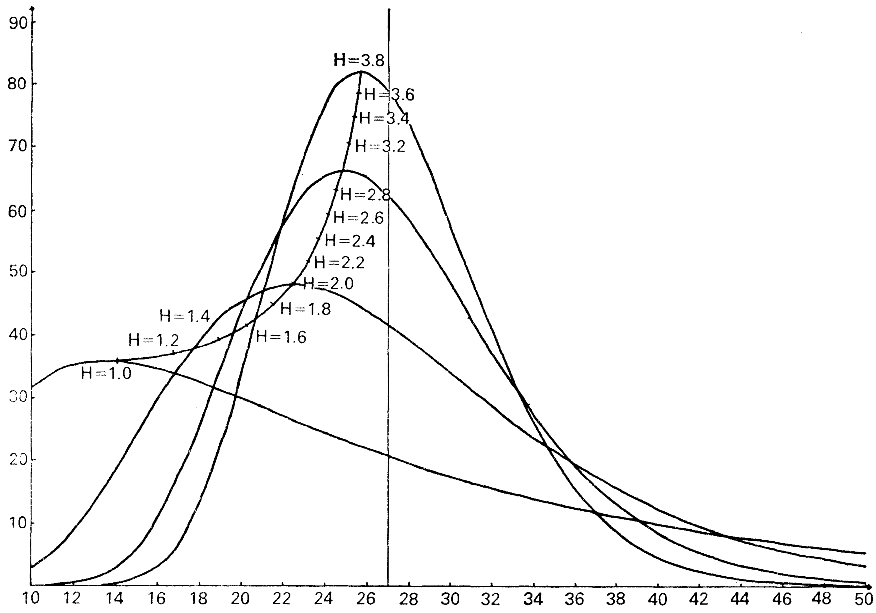

Fig. 1. Hadwiger curves and the locus of the mode corresponding to various values of H when $U=T=27$ $(D=0)$ and $R=1$.

below.) These "fits" are almost incredibly good, and we doubt whether it would be possible to choose one out of such a pair of functions if one were to graduate raw fertility rates with them.

On the other hand, the parameters R, U, M, and S^2 are quite stable (in the sense explained in Subsection 1 E) in the cases documented here as well as in a couple of other cases which we have studied. For these reasons, we recommend the replacement of the parameter vector (R, H, T, D) by (R, U, M, S^2).

2F. Each of the normed sums of squares of deviations in Table 2 has been calculated by dividing the straightforward sum of squares of deviations by the square of the mean of the corresponding two R-values. Thus, if the two curves are $R_1 f_1(x)$ and $R_2 f_2(x)$, and if $R_0=(R_1+R_2)/2$, then the corresponding normed sum of squares is

$$\sum_x \{R_1 f_1(x) - R_2 f_2(x)\}^2 / R_0^2.$$

The purpose of this procedure is to facilitate comparison of the fits between pairs of curves by eliminating the influence of the different areas under the various curves. Since $R_1 \simeq R_2$ in all the cases we consider, the normed sum of squares is roughly equal to

$$\sum_x \{f_1(x) - f_2(x)\}^2.$$

Table 1. *Some unconventional parameter values for the Hadwiger function which produce highly similar and reasonable fertility curves*

Curve no.	R (area)	H	T	D	$U=D+T$ (mean)	M (mode)	$S^2=T^2/(2H^2)$ (variance)
Group I							
1	0.974	5.478	48.149	− 22.107	26.042	24.854	38.628
2	0.983	14.267	125.420	− 99.766	25.654	25.193	38.640
3	0.983	51.026	443.415	−418.107	25.308	25.181	37.758
Group II							
4	1.061	51.891	515.445	−490.032	25.413	25.271	49.334
5	1.056	63.790	631.566	−605.836	25.730	25.603	49.012
6	1.049	74.961	749.431	−723.925	25.506	25.405	49.976

3. The inconsistency of moment estimators

3A. In a series of papers, Yntema (1953, 1956, 1969; Gilje & Yntema, 1971) has discussed estimators for the parameters R, H, T, and D of the Hadwiger function. One form of his estimators, as given by Hoem (1972, Subsection 7.4), is as follows.

Denote the "raw" observed fertility rate at age x by $\hat{\lambda}_x$. Let

$$\hat{R}=\sum_x \hat{\lambda}_x,\ \hat{U}=\sum_x x\hat{\lambda}_x/\hat{R},\quad \text{and } \tilde{M}=\min\{x: \hat{\lambda}_x \geqslant \hat{\lambda}_y \quad \text{for all } y\}.$$

Let $[y]$ denote the integer value of y, and let

$$\hat{V}=[\hat{U}+\tfrac{1}{2}],\ \hat{h}=\hat{\lambda}_{\hat{V}},\quad \hat{T}=\hat{R}^2/\{\tfrac{4}{3}\pi(\hat{U}-\tilde{M})\hat{h}^2\},$$

$$\hat{D}=\hat{U}-\hat{T},\quad \text{and } \hat{H}=\hat{h}\hat{T}\sqrt{\pi}/\hat{R}.$$

Then $(\hat{R}, \hat{H}, \hat{T}, \hat{D})$ are Yntema's estimators of (R, H, T, D). (Yntema, 1969, actually suggests some simple preliminary smoothing of the raw rates before $\tilde{M}$ and $\hat{h}$ are calculated.) In view of (2.3), (2.4) and (2.6), we let

$$\hat{M}=\hat{D}+\hat{T}\frac{(1+16\hat{H}^4/9)^{1/2}-1}{4\hat{H}^2/3} \tag{3.1}$$

Table 2. *Normed[a] sums of squares of deviations between pairs of curves in each group in Table 1, per 100 000*

Curve no.	Curve no. 2	3	5	6
1	3.93	9.41		
2		2.75		
4			4.07	1.46
5				2.34

[a] See Subsection 2F.

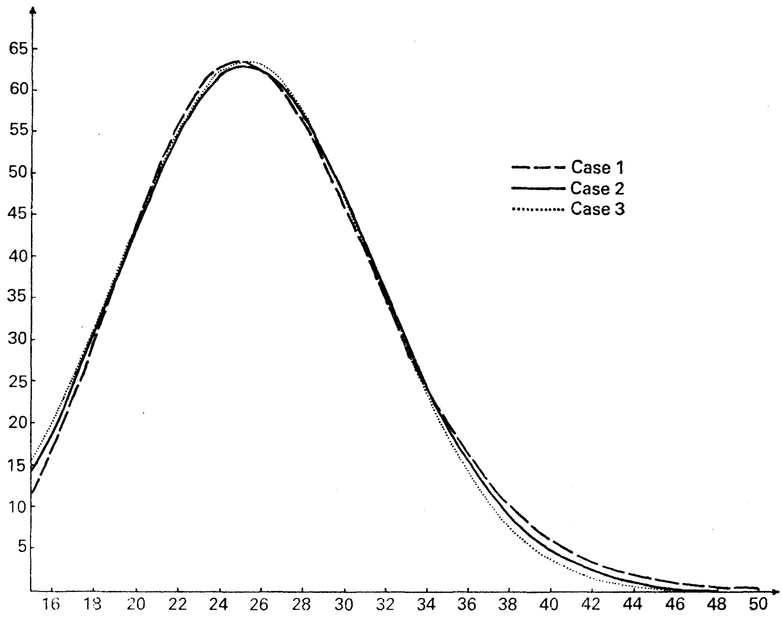

Fig. 2. Similar Hadwiger curves from different parameter sets (Table 1).

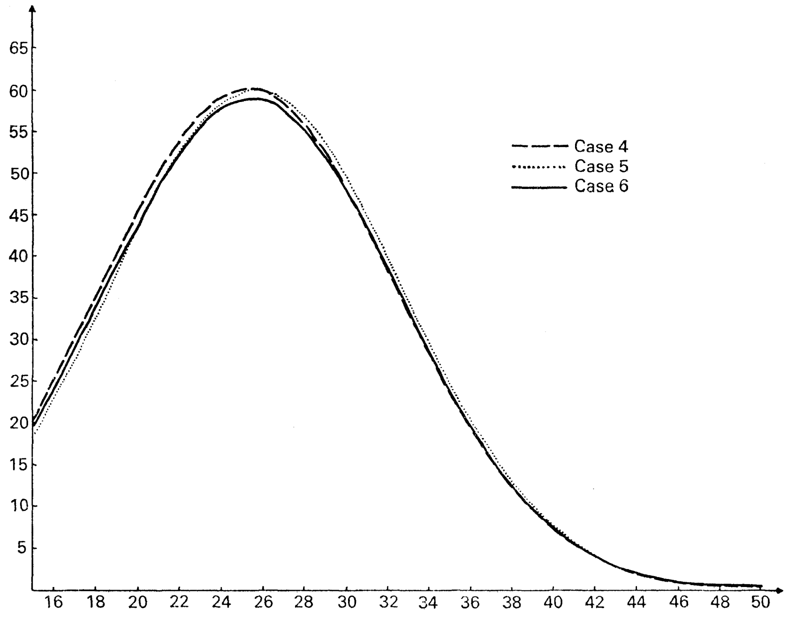

Fig. 3. Similar Hadwiger curves from different parameter sets (Table 1).

and

$$\hat{S}^2 = \frac{\hat{T}^2}{2\hat{H}^2} = \frac{\hat{R}^2}{2\pi\hat{h}^2}. \tag{3.2}$$

Following Yntema's lead, we have then used $(\hat{R}, \hat{U}, \hat{M}, \hat{S}^2)$ as moment estimators for (R, U, M, S^2). Thus, we regard $\tilde{M}$ only as a preliminary estimator for M, and substitute $\hat{M}$ for it when it has served its function.

3B. As the population size increases without bounds, $\hat{R}$, $\hat{H}$, $\hat{T}$, $\hat{D}$, $\hat{U}$, $\hat{M}$, and $\hat{S}^2$ converge in probability to values R_0, H_0, T_0, D_0, U_0, M_0, and S_0^2, respectively, and these values need not coincide with those of the parameters R, H, T, D, U, M, and S^2 which one wants to estimate. Formulas for R_0, H_0, T_0, D_0, U_0, M_0, amd S_0^2 are easily derived. Those for R_0, H_0, T_0, D_0, and U_0 are given by Hoem (1972, page 590) and will not be repeated here. The formulas for M_0 and S_0^2 are the same as (3.1) and (3.2) with $\hat{R}$, $\hat{H}$, $\hat{T}$, and $\hat{D}$ replaced by R_0, H_0, T_0, and D_0, respectively.

Previously, no one seems to have looked into the discrepancy between the R_0, H_0, etc., and the underlying parameters R, H, etc. We have listed a few comparisons in Table 3. It will be seen that we have used only two different sets of values for the underlying parameters. For both sets, we have applied Yntema's approach, with single year age groups between ages 15 and 50 (cases 1 and 2 in Table 3). In addition, we have applied his approach once more to the second set, again with single year age groups, but now with an age span from 15 to 44 (case 3).

The discrepancy between the quantities H_0, T_0, and D_0 on the one hand, and the corresponding parameters on the other hand, can be quite considerable. A comparison of cases 2 and 3 shows that even the choice of age groups can be important.

There is a corresponding discrepancy for R_0, U_0, M_0, and S_0^2, but it seems less important (at least for R_0, U_0, and M_0) than for H_0, T_0, and D_0.

3C. The formulas of Yntema's estimators have been set up by a largely intuitive argument which exploits an analogy between on the one hand the theoretical Hadwiger model, where age appears as a continuous variable, and on the other hand the real data, which are organized by age groups. The discrepancy between the probability limits of the estimators and the parameters estimated is due to the fixed discretization involved in the presentation of the real data by (say) single-year age groups. If the length of the age interval of each age group were permitted to decrease suitably to zero as the population size increases, and if the number of age groups were permitted to increase correspondingly, the discrepancy would disappear. This is not part of the kind of approach inherent in methods of analytic graduation of vital rates, however, and this type of discrepancy therefore becomes an integral part of graduation theory. It is not particular to Yntema's estimators, and we demonstrate it again in connection with the gamma density below.

Table 3. *Three sets of values of parameters R, H, T, D, U, M, and S^2 of the Hadwiger function, and the corresponding values of R_0, H_0, T_0, D_0, U_0, M_0, and S_0^2*

Note that cases 2 and 2* are based on the same data

Case no.		R	H	T	D	U	M	S^2	Single year age groups used
1	Parameter value[a]	3.716	1.206	15.440	14.260	29.700	23.669	81.967	15–50
	Yntema value[b]	3.589	1.864	21.571	7.084	28.655	24.497	66.946	
2	Parameter value[c]	2.957	1.764	18.239	9.263	27.502	23.628	53.453	15–50
	Yntema value[b]	2.925	2.293	22.549	4.666	27.215	24.228	48.336	
2*	Parameter value[c]	2.957	1.764	18.239	9.263	27.502	23.628	53.453	15–44
	Yntema value[b]	2.869	2.556	24.647	2.183	26.831	24.162	46.492	

[a] Values of *R, H, T*, etc. Source of values: Combined data for 1968–71 for 13 municipalities on the Norwegian West Coast (Kvitsøy, Bokn, Utsira, Austevoll, Sund, Øygarden, Austrheim, Fedje, Solund, Askvoll, Selje, Sande, and Giske).
[b] Values of R_0, H_0, T_0, etc. Derived by Yntema's method.
[c] Values of *R, H, T*, etc. Source of values: Gilje, 1969, p. 130 (Norway, 1966).

3D. Fortunately, the type of discrepancy demonstrated above does not seem to have important consequences for the results of analytic graduation by methods like, say, least squares. At least this is so for the numerical cases we have studied. We have proceeded as follows.

For each of the sets of values of *R, H, T*, and *D*, in cases 1 and 2 in Table 3, we have calculated the values of the Hadwiger function for ages 15, 16, ..., 50. We have then treated this set of function values, say $\lambda_{15}^*, \lambda_{16}^*, ..., \lambda_{50}^*$, as a set of "observed" fertility rates, we have calculated the corresponding Yntema "estimates", and we have finally used these as starting values for an iterative algorithm producing least squares "estimates". (Since there is nothing here that corresponds to the size of each population group, minimum chi-square "graduation" is out of the question.)

The results are impressive. The differences between the parameter values and their least squares "estimates" never exceed 1/1 000, and the sums of squares of deviations between the λ_x^* and *their* least squares estimates are of the size order of 10^{-6} of less. Such differences may well be due mostly to rounding errors.

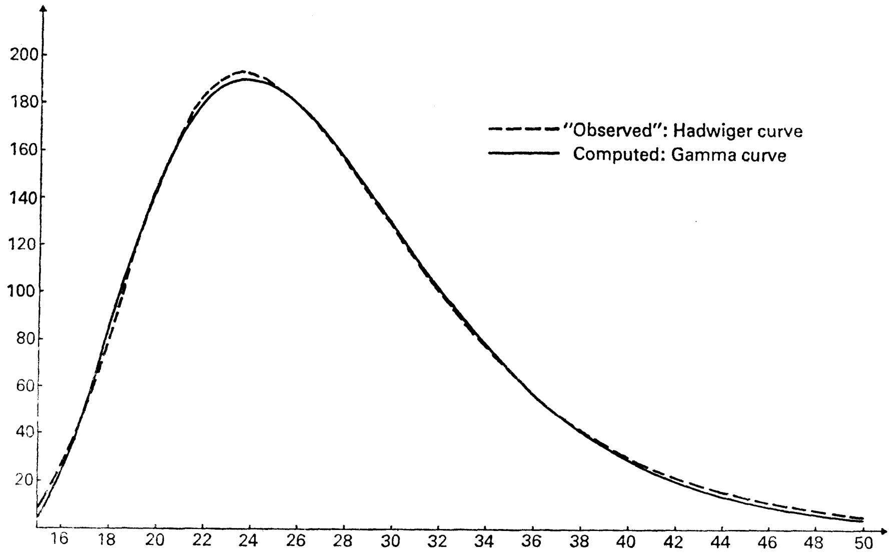

Fig. 4. Gamma graduation of Hadwiger curve for Norway, 1966 (case 2 of Table 5).

4. Graduation by means of the gamma density

4A. For comparison with Hadwiger graduation, we shall take a brief look at fertility graduation by means of the gamma density. In the latter case, the function which is fitted to the empirical fertility curve has the following form:

$$g(x; R, b, c, d) = \frac{R}{\Gamma(b)\, c^b}(x-d)^{b-1} e^{-(x-d)/c} \quad \text{for } x > d. \tag{4.1}$$

The parameter R still represents the level of fertility, and childbearing starts roughly at age d. If we denote the mean and modal ages of childbearing by μ and m, respectively, and the corresponding variance by σ^2, then

$c = \mu - m$, and $b = (\mu - d)/c = \sigma^2/c^2$.

Thus, all four parameter R, b, c and d are reasonably interpretable in terms of descriptive characteristics of the fertility curve. It is easy to construct moment estimators. It is also easy to reparametrize, e.g., by substituting, say, R, μ, m, and σ^2 for the four parameters used in (4.1).

4B. We have carried out some numerical experiments with the gamma density similar to the ones which we described in Subsection 3D for the Hadwiger function. Thus, we have constructed sets of "observed" fertility rates for a couple of cases, we have calculated corresponding moment "estimates", and we have used these as starting values for an iterative procedure which produces least squares "estimates". The results are roughly as encouraging as were the ones reported in Subsection 3D. For the cases which we have investigated, a parametrization by R, μ, m, and σ^2 turns out to give much more rapid conver-

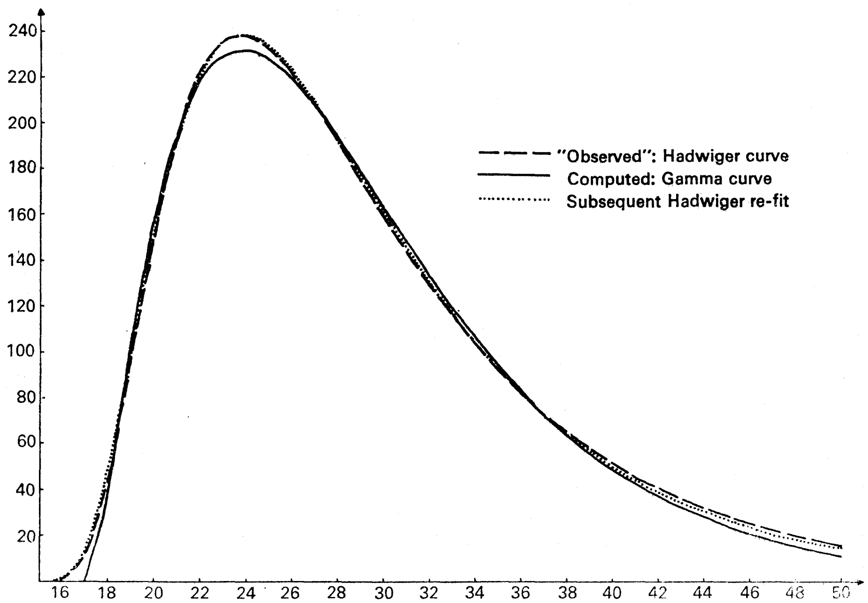

Fig. 5. Gamma graduation of Hadwiger curve and subsequent Hadwiger graduation of computed gamma curve for 13 Norwegian municipalities, 1968–71 (case 1 of Table 5).

gence than, say, the parametrization in (4.1). This is probably due to the greater stability of R, μ, m, and σ^2.

4C. In order to find out how well the gamma density function can represent a Hadwiger curve, we have carried out a gamma least squares "graduation" of the "observed" fertility curve corresponding to the Hadwiger functions with the parameter values of cases 1 and 2 of Table 3. The fitted parameter values of the gamma densities have been listed in Table 5, along with the normed sums of squares of deviations. The curves for case 2 have been plotted in Fig. 4. The fit is very good in this case. The curves for case 1 have been plotted as the stipled curve and the unbroken curve in Fig. 5. The fit is good, but not overwhelmingly so. Out of curiosity, we have subsequently fitted a second Hadwiger curve to the gamma curve of case 1. The resulting parameter values have been listed in parentheses as the "subsequent Hadwiger re-fit" in line 3 of case 1 of Table 5, and the corresponding curve is the dotted one in Fig. 5. This fit gets slightly better. It is interesting to see that the re-fit leaves us roughly where we started.

4D. We have gone on to fit a Hadwiger density by least squares to the "observed" fertility curves corresponding to the gamma density function with the parameter values of cases 2 and 3 of Table 4. The fitted parameter values have been listed in Table 6. The diagrams of these curves look quite similar to Fig. 4 without its stipled curve and are not displayed here. The fit is very good in both cases.

Table 4. *Three sets of values of parameters of the gamma density function, and corresponding probability limits of the moment estimators*

Case no.		R	μ	m	σ^2	b	c	d
1	Parameter value[a]	3.618	29.050	23.773	61.159	2.196	5.277	17.460
	Probability limit	3.553	28.517	25.685	46.924	5.851	2.832	11.948
2	Parameter value[b]	2.939	27.390	23.630	49.056	3.471	3.759	14.341
	Probability limit	2.918	27.188	24.588	43.588	6.448	2.600	10.422
3	Parameter value[c]	1.927	25.120	21.956	32.026	3.198	3.165	15.000
	Probability limit	1.925	25.080	22.314	30.882	4.038	2.766	13.914

[a] Thirteen Norwegian municipalities. (Corresponds to case 1 of Table 3.)
[b] Norway, 1966. (Corresponds to case 2 of Table 3.)
[c] Hungary, 1961. (Gilje, 1969, p. 134.)

4E. To the extent that the results reported in Subsection 4C and 4D carry over to other numerical examples, they raise an important question of principle. If these pairs of curves had appeared as the results of parallel graduations of real data, once by means of the Hadwiger function and once by the gamma density, we know of no objective way of selecting one out of the pair as more appropriate a graduation than the other. In other words, it looks as if we are

Table 5. *Parameter values of the gamma density fitted to the Hadwiger function of cases 1 and 2 in Table 3*

		R	Mean	Mode	Variance	Normed sum of squares of deviations[d]
Case 1	Hadwiger parameter[a]	3.716	29.700	23.669	81.967	
	Gamma parameter	3.618	29.050	23.773	61.159	$4.2\cdot 10^{-5}$
	(Subsequent Hadwiger re-fit)[b]	(3.695)	(29.460)	(23.792)	(76.001)	$(3.0\cdot 10^{-5})$
Case 2	Hadwiger parameter[c]	2.957	27.502	23.628	53.453	
	Gamma parameter	2.928	27.322	23.687	48.254	$1.0\cdot 10^{-5}$

[a] Thirteen Norwegian municipalities, 1968–71 (case 1 of Table 3).
[b] For an explanation, see Subsection 4C.
[c] Norway, 1966 (case 2 of Table 3).
[d] For an explanation of the norming procedure, see Subsection 2F.

Table 6. *Parameter values of the Hadwiger density fitted to the gamma densities of cases 2 and 3 in Table 4*

Case no.[a]	R	Mean	Mode	Variance	Normed sum of squares of deviations[d]
2 Gamma parameter[b]	2.939	27.390	23.630	49.056	
Hadwiger parameter	2.967	27.546	23.603	54.069	$1.2 \cdot 10^{-5}$
3 Gamma parameter[c]	1.927	25.120	21.956	32.026	
Hadwiger parameter	1.947	25.258	21.937	35.654	$1.9 \cdot 10^{-5}$

[a] For case 1, see Table 5 and Subsection 4C.
[b] Norway, 1966.
[c] Hungary, 1961.
[d] For an explanation of the norming procedure, see Subsection 2F.

faced with a second practical identification problem connected with the Hadwiger function, over and above the one reported in Subsection 2E above. The second problem is that of distinguishing the values of the Hadwiger function from corresponding values of a gamma density in a manner sufficiently clear to be of practical use in applications to real data.

This finding fits well in with the fact that previous research has failed to establish either of these functions as systematically superior to the other for purposes of fertility graduation.

References

1. Berge, E. & Hoem, J. M. (1974). Nokre praktiske røynsler med analytisk glatting. Central Bureau of Statistics of Norway, Working Paper IO 74/23. To appear in the *Statistical Review* of the Swedish National Central Bureau of Statistics.
2. Brass, W. et al. (1968). *The Demography of Tropical Africa*, p. 109. Princeton University Press.
3. Farid, S. M. (1973). On the pattern of cohort fertility. *Population Studies* **27**, 159–168.
4. Gilje, E. (1969). Fitting curves to age-specific fertility rates: Some examples. *Statistical Review* of the Swedish National Central Bureau of Statistics, III, 7 (2): 118–134. Reprinted in Article No. 51 from the Central Bureau of Statistics of Norway, 1972.
5. Gilje, E. & Yntema, L. (1971). The shifted Hadwiger fertility function. *Skand. Aktuarietidskr.* **54**, 4–13. Reprinted in Article No. 51 from the Central Bureau of Statistics of Norway, 1972.
6. Gruveaus, G. T. & Jöreskog, K. G. (1970). A computer program for minimizing a function of several variables. Educational Testing Service, Princeton, N.J., *Research Bulletin* 70–14.
7. Hadwiger, H. (1940). Eine analytische Reproduktionsfunction für biologische Gesamtheiten. *Skand. Aktuarietidskr.* **23**, 101–113.
8. Hadwiger, H. & Ruchti, W. (1941). Darstellung der Fruchtbarkeit durch eine biologische Reproduktionsformel. *Archiv für Mathematische Wirtschafts- und Sozialforschung* **7**, 30–40.
9. Hoem, J. M. (1971). On the interpretation of the maternity function as a probability density. *Theor. Popul. Biol.* **2**, 319–327. Correction Note: **3**, 240.
10. Hoem, J. M. (1972). On the statistical theory of analytic graduation. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **1**, 569–600. Reprinted as Article No. 49 from the Central Bureau of Statistics of Norway, 1972.
11. Hoem, J. M. et al. (1974). Two papers on analytic graduation. Central Bureau of Statistics of Norway, Working Paper IO 74/17. To appear in *Scand. J. Statist.* and *Proc. 8th Biometrics Conf., Constanta, Romania, August, 1974.*

12. Hunyadi, L. & Szakolczai, G. (1970). Distribution curves of the age-specific probabilities of birth and death and their shift by time. (In Hungarian.) *Demográfia* **13**, 242–268.
13. Hyrenius, H., Sundvall, A. & Nygren, O. (1974). Methods for fitting a Pearson type I function to age-specific fertility rates. *Statistical Review* of the Swedish National Central Bureau of Statistics, III, 12 (2): 133–142.
14. Mitra, S. (1970). Graduation of net maternity function. *Sankhya Ser. B* **32**, 63–68.
15. Mitra, S. & Romaniuk, A. (1973). Pearsonian type I curve and its fertility projection potentials. *Demography* **10**, 351–356.
16. Murphy, E. M. & Nagnur, D. N. (1972). A Gompertz fit that fits: Applications to Canadian fertility patterns. *Demography* **9**, 35–50.
17. Nelder, J. A. & Mead, R. (1965). A simplex method for function minimization. *Computer Journal* **7**, 308–313.
18. Norman, K. & Gilje, E. (1973). Analytic graduation of age specific fertility rates in the London boroughs. Greater London Council, Department of Planning and Transportation, Intelligence Unit, Research Memorandum No. 423.
19. O'Neill, R. (1971). Algorithm AS 47. Function minimization using a simplex procedure. *Applied Statistics* **20**, 338–345.
20. Romaniuk, A. (1973). A three parameter model for birth projections. *Popul. Studies* **27**, 467–478.
21. Tekse, K. (1967). On demographic models of age-specific fertility rates. *Statistical Review* of the Swedish National Central Bureau of Statistics, III, 5 (3): 189–207.
22. Yntema, L. (1952). *Mathematical models of demographic analysis.* J. J. Groen & Zoon, Leiden.
23. Yntema, L. (1953). The graduation of net fertility tables. *Boletim do Instituto dos Actuários Portogueses* **8**, 29–43.
24. Yntema, L. (1956). An approximation of the family structure. *First Internat. Conf. of Social Security Actuaries and Statisticians, Brussels,* **3**: 351–366.
25. Yntema, L. (1961). Demometric invariants. *Actuariële Studiën* **3**, 93*–101*.
26. Yntema, L. (1969). On Hadwiger's fertility function. *Statistical Review* of the Swedish National Central Bureau of Statistics, III, 7 (2): 113–117.

Addendum

Ole Barndorff-Nielsen and Søren Johansen have brought to our attention the fact that the Hadwiger density discussed in Section 2 [above] has appeared in the statistical literature before. Johnson & Kotz (1970) devote a chapter called "Inverse Gaussian (Wald) distributions" to it, and later papers by Wani & Kabe (1973) and Basu & Wasan (1974) contain further material.

Basu, A. K. & Wasan, M. T. (1974). On the first passage time processes of Brownian motion with positive drift. *Scand. Actuarial J.* 1974, 144–150.
Hoem, J. M. & Berge, E. (1975). Some problems in Hadwiger fertility graduation. *Scand. Acturial J.* 1975.
Johnson, N. L. & Kotz, S. (1970). *Continuous Univariate Distributions, Vol. 1.* Houghton Mifflin Co., Boston.
Wani, J. K. & Kabe, D. G. (1973). Some results for the inverse Gaussian process. *Skand. Aktuarietidskrift* 1973, 52–57.

Received October 1974

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